IR finiteness of the ghost dressing function from numerical resolution of the ghost SD equation

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ABSTRACT: We solve numerically the Schwinger-Dyson ghost equation in the Landau gauge for a given, finite at $k = 0$ gluon propagator (i.e. the infrared exponent of its dressing function, α_{gluon}, is 1) and under the usual assumption of constancy of the ghost-gluon vertex ; we show that there exist two possible types of ghost dressing function solutions, as we have previously inferred from analytical considerations: one which is singular at zero momentum (the infrared exponent of its dressing function, α_{ghost},[†]is < 0), satisfies the familiar relation $\alpha_{\mathrm{gluon}} + 2\alpha_{\mathrm{ghost}} = 0$ and has therefore $\alpha_{\mathrm{ghost}} = -1/2$, and another one which is finite at the origin with $\alpha_{\mathrm{ghost}} = 0$ and violates the relation. It is most important that the type of solution which is realized depends on the value of the coupling constant. There are regular ones — $\alpha_F = 0$ — for any coupling below some value, while there is only one singular solution — $\alpha_F < 0$ —, obtained for a single critical value of the coupling. For all momenta $k < 1.5\,\mathrm{GeV}$ where they can be trusted, our lattice data exclude neatly the singular one, and agree very well with the regular solution we obtain at a coupling constant compatible with the bare lattice value.

KEYWORDS: Lattice Gauge Field Theories, Lattice QCD, QCD.

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[†]We shall use α_G and α_F as shorthands for $\alpha_{\mathrm{gluon}}$and α_{ghost} respectively; let us recall that we denote the gluon by a G and the ghost by a F, for "fantôme".

Contents

1. Introduction

Since the first attempts, an impressive progress has been made in understanding the solutions to the Schwinger-Dyson (SD herafter) equations for the QCD propagators and their behaviour at small momenta. In particular, an important step has been accomplished by putting forward the essential contribution of internal ghost loops in the gluon propagator equation, previously neglected; it has been shown that it may completely change the previously expected behavior of the gluon propagator from much more singular than the free one (diverging like $1/k^4$- as was believed for a long time), to being much less singular [1].[1] All the following considerations are made assuming the choice of the Landau (i.e. Lorentz) gauge.

The consensus before 2005. For some years, a consensus seemed to be obtained around a statement of 1) an infrared fixed point of the gluon-ghost coupling and 2) a singular ghost dressing function (see below for more explanation). This consensus was very strong and unopposed, since several other approaches were apparently converging to the same conclusions. Such authoritative people as S. Brodsky have also appealed to it to support their considerations about AdS/CFT (see [2][2]). Another part of the consensus, deduced from a solution to coupled SD equations, was the statement that $\alpha_G > 1$, i.e. the gluon propagator should necessarily vanish; yet this was contested by a thorough calculation of Bloch [3].

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[1] For this particular case, see especially the section 3 of the quoted paper.

[2] See especially section 3 of the paper by Brodsky giving references to certain lattice data and to non perturbative statements like solutions of SD equations.

To be more specific about this consensus, a usual assumption for the infrared behaviour of gluon and ghost dressing functions is that they should be power behaved, i.e. for the gluon $G(k)$ and the ghost $F(k)$ *dressing functions* respectively:

$$G(k) \sim (k^2)^{\alpha_G}, \tag{1.1}$$

$$F(k) \sim (k^2)^{\alpha_F}. \tag{1.2}$$

In a number of studies [1, 4], it has been stated that for a suitably simple assumption concerning the ghost-gluon and gluon vertices, the SD coupled equations for $G(p)$ and $F(p)$ imply

$$\alpha_G + 2\alpha_F = 0. \tag{1.3}$$

This statement is the starting point for the popular claim of an infrared fixed point for the QCD renormalised coupling constant. In fact, admitting the validity of eq. (1.3), the IR fixed point would be present in the coupling constant defined by the ghost-gluon 3 points Green function in a MOM scheme. Let us recall the renormalisation conventions, with bare quantities denoted by a "B" subindex. In general:

$$G_B(k^2) = Z_3 \, G_R(k^2), \qquad F_B(k^2) = \widetilde{Z}_3 \, F_R(k^2), \qquad g_B = Z_g \, g_R,$$
$$\Gamma_R = \widetilde{z}_1 \, \Gamma_B, \qquad Z_g = \widetilde{z}_1 \, (Z_3^{1/2} \, \widetilde{Z}_3)^{-1}. \tag{1.4}$$

In the MOM schemes:

$$Z_3 = G_B(\mu^2), \tag{1.5}$$

$$\widetilde{Z}_3 = F_B(\mu^2), \tag{1.6}$$

$$G_R(k^2, \mu) = G_B(k^2)/G_B(\mu^2), \tag{1.7}$$

$$F_R(k^2, \mu) = F_B(k^2)/F_B(\mu^2), \tag{1.8}$$

while many possibilities are open for the renormalisation condition of the vertex. We need not specify it for reasons explained hereafter below eq. (2.1). Then:

$$g_R(\mu) = g_B G_B(\mu^2)^{1/2} F_B(\mu^2)/\widetilde{z}_1(\mu). \tag{1.9}$$

This implies that the product $g_R(\mu)\widetilde{z}_1(\mu)(G_R(k^2, \mu))^{1/2}F_R(k^2, \mu)$ is independent of μ.

Now, $g_R(\mu)$ would tend to a finite limit for small μ if eq. (1.3) would hold, under the additional assumption that $\widetilde{z}_1(\mu)$ is finite for $\mu \to 0$.[3]

The input of lattice data. Recently, lattice data have also entered the game and have contributed much to the discussion, by showing features quite contrary to this consensus. Our motivation here is to try to clarify the situation within the SD approach by exhibiting

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[3]Note that the UV finiteness of $\widetilde{z}_1(\mu)$ does not imply that it is μ independent, even in perturbation; see in particular, for the symmetric MOM scheme, eq. (13) in our ref. [5], extracted from the results of Chetyrkin and Retey [6]; however, we can suppose that the non perturbative IR behavior is not too singular.

new numerical solutions restoring the agreement between the lattice data, the numerical study in the continuum and the analytical considerations.[4]

We can test relation (1.3) on the lattice by computing the quantity $G(k)F(k)^2$ which, according to it, would be expected to tend to a finite value at small k. In fact this is clearly contradicted by the lattice data, as can be seen on figure 1 of our paper [5] and, in the work of Sternbeck and collaborators, on figure 4 of [9] or figure 3 of [10], all of which show that the product **decreases rapidly** at small k, possibly to zero. In addition, as we will see in sections 2 and 3, for solutions satisfying the relation (1.3) the value to which the product $N_c g_R{}^2(\mu)\widetilde{z}_1(\mu)G_R(k^2,\mu)F_R{}^2(k^2,\mu)$ must tend when $k \to 0$ is much larger than the value observed at the smallest accessible momenta (it should be $10\,\pi^2$ in the case $\alpha_G = 1$, assuming the renormalized ghost-gluon vertex function $H_{1R}(q,k,\mu)$ to be equal to 1).

In the above studies, which try to solve the **coupled** SD equations, the gluon propagator is also predicted, and it is found that α_G is positive, which is anyway also suggested by lattice QCD. If so, and if the relation (1.3) would hold, then it would imply finally that $\alpha_F < 0$, i.e.that the ghost dressing function should be singular. This statement would be in agreement with the Kugo-Ojima criterion for confinement, a convergence which would make the general picture obtained theoretically appealing.

However, from the admitted values of $\alpha_G \gtrsim 1$, $\alpha_F \lesssim -1/2$, $F(k^2)$ should present a power behavior close to $1/k$, or even more singular. This stronger result is excluded by the lattice data, which allow at most a **very weak** singularity ; indeed according to our first analysis [5] and to the study of Sternbeck et al. [11] (see also their recent large volume study [12]), the power seems to be at most $\alpha_F = -0.2$ down to momenta around $0.3\,\mathrm{GeV}$;[5] in fact, we have obtained better fits of our own SU(2) and SU(3) data with logs rather than with powers [13]; finally, if we abandon any prejudice, it appears that it is compatible with a finite value as well. Certainly, a value close to $\alpha_F = -0.5$ is not possible, unless there is a sudden change of behavior very near $k = 0$. This conclusion is reinforced by the recent results of Cucchieri and Mendes at very large volumes [14].

Analytical setting. In view of this situation, in our paper [5], we started a new discussion on the implications of the ghost SD equation for the IR behavior of the ghost propagator.

We consider this equation in its **subtracted, UV convergent** form:

$$\frac{1}{F_B(k^2)} - \frac{1}{F_B(k'^2)} = -N_c g_B^2 \int \frac{d^4q}{(2\pi)^4} \left(1 - \frac{(k.q)^2}{k^2 q^2}\right) \times \tag{1.10}$$

$$\times \left[\frac{G_B((q-k)^2)H_{1B}(q,k)}{((q-k)^2)^2} - \frac{G_B((q-k')^2)H_{1B}(q,k')}{((q-k')^2)^2}\right] F_B(q^2)$$

with k' an arbitrary subtraction point, taken for simplicity parallel to k, $k' = k\sqrt{k'^2/k^2}$.

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[4]An attempt to describe the lattice data within SD coupled equations is made in [7]. For a recent attempt to accommodate the lattice data (with a finite non zero ghost dressing function) within the Gribov-Zwanziger approach see [8].

[5]Note that these authors plot q^2 along the x axis in the figures.

H_1 is one of the invariants in the Lorentz decomposition of the ghost-gluon vertex:

$$\widetilde{\Gamma}_{B\mu}^{abc}(-q,k;q-k) = ig_B f^{abc} q_\nu \widetilde{\Gamma}_{B\nu\mu}(-q,k;q-k)$$
$$= ig_B f^{abc}(q_\mu H_{1B}(q,k) + (q-k)_\mu H_{2B}(q,k)) \tag{1.11}$$

where $-q$, k and $q-k$ are respectively the entering momenta of the outgoing ghost, the ingoing one and the gluon.

In all our present considerations this equation is considered with given gluon propagator and vertex as *ansätze*, and the ghost dressing function appears then as the solution to the equation. This is what we call the SD ghost equation. We do not try to solve any other SD equation. The advantage of concentrating on this equation is that it is much simpler than the gluon one or any other, to the point that analytical statements can be formulated for the ghost for any given gluon propagator and vertex. On the other hand, various assumptions on the IR behavior of the gluon propagator and the vertex may be used, in particular those advocated in the above references. Both these inputs and the output ghost solution can be tested by means of a comparison with the lattice data.

Using the definitions given in eq. (1) the renormalised form of this equation is obtained as:

$$\frac{1}{F_R(k^2)} - \frac{1}{F_R(k'^2)} = -N_c g_R^2 \widetilde{z}_1 \int \frac{d^4q}{(2\pi)^4} \left(1 - \frac{(k.q)^2}{k^2 q^2}\right) \times \tag{1.12}$$
$$\times \left[\frac{G_R((q-k)^2)H_{1R}(q,k)}{((q-k)^2)^2} - \frac{G_R((q-k')^2)H_{1R}(q,k')}{((q-k')^2)^2}\right] F_R(q^2).$$

We know that, in Landau gauge, $H_{1B}(q,0) + H_{2B}(q,0) = 1$ which implies that $\widetilde{z}_1$ is finite for any momentum configuration.[6] Let us remark that this implies that the subtracted SD equation is convergent. Indeed, $\widetilde{z}_1$ and the l.h.s. of eq. (1.12) being finite the integral in the r.h.s. must be convergent. This was not obvious in the bare version.

In the following, we set $k'^2 = \mu^2$ to get the one variable renormalised integral equation.

One can wonder whether the solutions of this subtracted SD equation are also solutions of the unsubtracted one:

$$\frac{1}{F_R(k^2)} = \widetilde{Z}_3 - N_c g_R^2 \widetilde{z}_1 \int \frac{d^4q}{(2\pi)^4} \left(1 - \frac{(k.q)^2}{k^2 q^2}\right) \times$$
$$\times \left[\frac{G_R((q-k)^2)H_{1R}(q,k)}{((q-k)^2)^2}\right] F_R(q^2). \tag{1.13}$$

This is seen to hold simply by making:

$$\widetilde{Z}_3 = 1 + N_c g_R^2 \widetilde{z}_1 \times$$
$$\times \int \frac{d^4q}{(2\pi)^4} \left(1 - \frac{(k.q)^2}{k^2 q^2}\right) \left[\frac{G_R((q-k)^2)}{((q-k)^2)^2}\right] H_{1R}(q,k) \, F_R(q^2) \Big|_{k^2=\mu^2}. \tag{1.14}$$

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[6]Let us recall that what has been really demonstrated in the paper of Taylor [15] is the equation we have just written, i.e. for a vertex with zero ingoing ghost momentum. Then $\widetilde{z}_1 = 1$ for this particular MOM renormalisation; in general it will remain finite but different from 1. The detailed explanations on the Taylor paper are given in our article [5].

Of course, one has now to regularise the integral in some way, and this introduces a finite arbitrariness in $\widetilde{Z}_3$. The divergence of the integral in equation (1.14), which is responsible for the divergence of $\widetilde{Z}_3$ itself, will be of course cancelled, upon insertion in eq. (1.13), by the second integral, similarly to what occurred in the subtracted form (1.12).

Introducing the regular solutions for the ghost dressing function. We concluded in ref. [5] that, **in general**, under the usual IR regularity assumption for the ghost-gluon vertex, the SD ghost equation implies the relation (1.3) by itself, without recoursing to the gluon equation. There were exceptions however (see below), but we first discarded them. Therefore, since the relation is definitely seen to be violated on the lattice, while the SD equation is automatically satisfied, we first suggested in the same paper, as a way out of this puzzle, that the vertex invariant H_1 could be IR singular instead of being constant.

Then, it soon appeared, in view of the lattice data, in particular thanks to Sternbeck et al. [10] as well as to the previous work of Cucchieri et al. [16], that this possibility is very unprobable: indeed they measure $H_{1B}(q, q)$ (gluon at zero momentum, and contraction with q_μ), and they find it roughly constant and close to 1. Therefore, our attention has been drawn to the cases, predicted in our analytical discussion of the SD ghost equation [5], where the relation 1.3 can be violated in spite of having a regular ghost-gluon vertex. These are the cases where $\alpha_G \simeq 1$ and $\alpha_F = 0$, i.e. where **the ghost dressing function is regular**[7] **at the origin**. As we have said, we did not pay attention to them in the beginning. But we have become aware that this possibility is attractive because:

1) on the lattice, α_G seems to be not far from 1, cf. refs. [17, 18], i.e. the gluon propagator is not far from being finite (see also the very recent very large volumes studies of the above references [12, 14]). Thus it automatically leads to a rapidly decreasing $G(k^2)F(k^2)^2$, $\mathcal{O}(k^2)$, behaviour when k approaches 0 in agreement with what is observed.

2) last but not least, on the lattice, the effective α_F is compatible with 0, as we have seen above.

Therefore, the appealing possibility $\alpha_F = 0$ has been adopted in our subsequent paper [13].

On the other hand, no statement can be deduced from these analytical considerations concerning the ghost SD equation as to which solution for the ghost propagator should be effectively preferred in real QCD. **A complementary theoretical input comes from the Slavnov-Taylor (ST) identity for the three-gluon vertex**. From this identity, we have demonstrated in [19] that the ghost dressing function should be IR finite.

The aim of the present paper is to reconsider this question by a **numerical** study of the ghost SD equation with input from lattice data for the gluon propagator and the simple and widely admitted constancy assumption for the vertex. The conclusion is striking: it is found that the IR finite solutions violating the relation (1.3) indeed exist. Which type of solution is realized, either singular or regular at $k \to 0$, depends on the actual value of the

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[7]For some qualification of the term "regular" used in the present context, see below after eq. (1.15).

QCD coupling constant. **One and only one** singular solution is obtained, for **only one** value of g^2 which we call "critical", and it can be completely exhibited; according to our calculation, **it cannot be the one of real QCD, because it disagrees grossly with the lattice results over a large range of momenta**. Therefore, in agreement with our ST statement, the actual ghost dressing function must be regular, i.e. IR finite, and indeed **we find solutions regular at** $k \to 0$ **describing very well the lattice ghost data**, with values of the coupling constant close to the one estimated from the actual bare coupling constant of the lattice. In summary, the combination of the numerical resolution and of the ghost lattice data or the ST theoretical input allows to discard the singular solution.

Warnings. A caveat must be made now. To make the discussion simple and to keep it close to the commonly accepted conceptual framework, we have adopted above the usual assumption of a **pure power** IR behaviour for dressing functions. But this is by no means a necessary assumption; especially for a massless theory, it would not be unexpected to have a behavior with log factors accompanying integer powers of k^2. For instance, according to our Slavnov-Taylor discussion [5, 19, 20], the gluon propagator must be infinite at the origin under some regularity assumption for the three-gluon vertex. Then a way to reconcile this statement with the observation of an apparently IR finite propagator from the lattice is to assume that this divergence is logarithmic, which would make it very difficult to detect on the lattice.[8] In this case, the behavior of $G(k)$ would be $G(k) \simeq k^2 (\log(k^2))^\nu$ ($\nu > 0$). We present the results disregarding the logs, although we have also checked in our numerical calculation that **including such a log in the gluon propagator does not change appreciably the ghost propagator** deduced from the SD equation.

In addition, in our analytical discussion [19], we have shown that if $\alpha_G = 1$ and $\alpha_F = 0$ one must have in the ghost dressing function logarithms of the type

$$F(k) = a + b \; k^2 \log(k^2). \tag{1.15}$$

The effect of such logarithms is very weak so that for the present purpose we qualify this as "regular", and anyway, it *is* IR finite. Nevertheless, it is possible to display the effect of this logarithm in the numerical calculation.

As we have just said, we have checked that including a log in the gluon propagator does not change appreciably the ghost propagator deduced from the SD equation. This is in agreement with an analytical argument which shows that only the power of the log in eq. (1.15) is changed.

Assumptions and inputs of the calculation. To summarise, our starting assumptions for the numerical calculation below are the following:

1) we take the ghost gluon vertex invariant $H_1(k, q)$ implied in the SD ghost equation (see below) as momentum independent. This is a rather usual assumption made in

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[8]Of course, the gluon propagator is always finite at $k = 0$ in a finite volume. What we mean here is that the gluon propagator at $k = 0$ seems to be relatively constant for increasing volume. But one cannot exclude a very slow variation, such as expected from a logarithm.

SD studies and, as we said above, it is in rough compatibility with present lattice data. In fact, it would be sufficient to assume simply a regular vertex to get the same qualitative conclusions, but for a numerical study we have to make a definite choice. Let us emphasize that our first goal is not to make a realistic quantitative prediction, but simply to demonstrate that, even with this type of regularity assumption, one can obtain solutions which do not obey $\alpha_G + 2\alpha_F = 0$, contrarily to the common belief. For this purpose, it is not necessary to bother about what would be the most realistic assumption for the vertex.

2) for the gluon propagator, in the small momentum region, we use an interpolation of the gluon propagator given by the lattice, with $\alpha_G = 1$.

We are aware that several SD studies, for example [1], exclude this latter possibility when considering the coupled equations, since α_G and α_F are then determined separately and it is found then that $\kappa = -\alpha_F = 1/2\,\alpha_G > 1/2$, implying therefore $\alpha_G > 1$. At this point, we recall that the paper by Bloch [3] concerning the coupled equations finds solutions with $\alpha_G = 1$ and $\alpha_G + 2\alpha_F = 0$ ($\kappa = 1/2$), thanks to a more refined treatment of the gluon SD equation. He can then reproduce rather well the gluon lattice propagator. Then the question is compelling: knowing that the gluon lattice data are well reproduced by him, and that the lattice data satisfy the ghost SD equation he is solving, how can it be that our lattice ghost dressing function exhibits an infrared behaviour different from his prediction $1/k$? We show that this is due not to lattice artefacts but to the possibility — neglected by him — that there be different types of solutions to the ghost equation for the **same** gluon propagator, depending on the value of the coupling constant.

2. Analytical considerations on the behaviour at small k

As a preliminary to the numerical study, let us recall or establish **analytical** relations which can be used as tests of the soundness and accuracy of our numerical calculation. We extract them from a more complete discussion which will be given in a forthcoming paper [21].

In this section and hereafter, since we adopt the constancy assumption for the ghost-gluon vertex, it appears immediately that the coupling constant only appears in the combination:

$$\widetilde{g}^2 \equiv N_c g_R^2 \widetilde{z}_1 \, H_{1R} = N_c g_B^2 (Z_3 \widetilde{Z}_3^2/\widetilde{z}_1) \, H_{1R} = N_c g_B^2 Z_3 \widetilde{Z}_3^2 \, H_{1B} \tag{2.1}$$

where H_{1R} or H_{1B} are constants. We use this auxiliary notation througout the rest of the article. From the last equality, it is obvious that $\widetilde{g}^2$ is independent of the way one renormalises the vertex since only the bare vertex appears.

We proceed as in our paper [5]:

- we separate the integral into a UV part and an IR part, characterised respectively by $q > q_0$ and $q < q_0$ for some suitably chosen q_0.

-for the infrared contribution (see eq. (14) and (17) of [3] with $\alpha_\Gamma = 0$ and H_1 and h constant), we use the power laws $F_R(k) \simeq A(k^2)^{\alpha_F}$ and $G_R(k) \simeq B(k^2)^{\alpha_G}$, assumed to hold as $k \to 0$, i.e. for $k < q_0$.

-we write eq. (1.12) replacing k by λk, taking $k' = \lambda \kappa k$ (κ fixed and < 1) and performing the change of variable $q \to \lambda q$. We then consider the IR limit $\lambda \to 0$; we show that, then, the UV part goes to zero at least as fast as λ^2.

1) **Singular solution**. Let us establish a relation for $\widetilde{g}_c^2$, the value of $\widetilde{g}^2$ corresponding to the singular solution, which is found to be unique. It is inspired by Bloch [3].

One can write, at leading order in λ:

$$(\lambda^2 k^2)^{-\alpha_F}(1 - \kappa^{-2\alpha_F}) \simeq -\widetilde{g}_c^{\,2} \, \lambda^{2(\alpha_F + \alpha_G)} \, A^2 B \int^{q < q_0/\lambda} \frac{d^4q}{(2\pi)^4} (q^2)^{\alpha_F} \qquad (2.2)$$

$$\times \left(1 - \frac{(k.q)^2}{k^2 q^2} \right) \left[((q-k)^2)^{\alpha_G - 2} - ((q - \kappa k)^2)^{\alpha_G - 2} \right].$$

For $\alpha_G = 1$ and $\alpha_F < 0$ this integral, being $\mathcal{O}(\lambda^{2(\alpha_F + \alpha_G)})$, dominates by a negative power of λ over the UV part which is $\mathcal{O}(\lambda^2)$. We can then neglect the UV part. This gives the relation $2\alpha_F + \alpha_G = 0$, whence $\alpha_F = -1/2$ and $F(k^2) \simeq 1/k$. Furthermore one can give an analytic expression for the IR integral in terms of the function:

$$f(a,b) = \frac{1}{16\pi^2} \frac{\Gamma(2+a)\Gamma(2+b)\Gamma(-a-b-2)}{\Gamma(-a)\Gamma(-b)\Gamma(4+a+b)}. \qquad (2.3)$$

Its value is equal to $(1 - \kappa^{-2\alpha_F})(k^2)^{-\alpha_F}\Phi(\alpha_G)$ with

$$\Phi(\alpha_G) = -\frac{1}{2}\left(f\left(-\frac{\alpha_G}{2}, \alpha_G - 2 \right) + f\left(-\frac{\alpha_G}{2}, \alpha_G - 1 \right) + f\left(-\frac{\alpha_G}{2} - 1, \alpha_G - 1 \right) \right) +$$

$$+ \frac{1}{4}\left(f\left(-\frac{\alpha_G}{2} - 1, \alpha_G - 2 \right) + f\left(-\frac{\alpha_G}{2} - 1, \alpha_G \right) + f\left(-\frac{\alpha_G}{2} + 1, \alpha_G - 2 \right) \right),$$

$$(2.4)$$

which leads to $\widetilde{g}_c^2 A^2 B = \frac{1}{\Phi(\alpha_G)}$. In our case, $\alpha_G = 1$ and $\alpha_F = -\frac{1}{2}$, $\Phi(1) = \frac{1}{10\pi^2}$ and the relation becomes:

$$\widetilde{g}_c^2 A^2 G_R^{(2)}(0) = 10\pi^2 \qquad (2.5)$$

where $G_R^{(2)}$, the gluon **propagator**, if finite at the origin.

This is only a relation between $\widetilde{g}_c^2$ and A and it does not allow us to know a priori the value of $\widetilde{g}_c^2$ before any numerical computation, unless a very small renormalization point μ is chosen. In this case we have: $A = (\mu^2)^{-\alpha_F}$ and $B = (\mu^2)^{-\alpha_G}$ so that $\widetilde{g}_c^2 = \frac{1}{\Phi(\alpha_G)}$ ($\widetilde{g}_c^2 = 10\pi^2$ when $\alpha_G = 1$).

2) **Regular solutions.** If, on the other hand, $\alpha_F = 0$ the l.h.s. is trivially zero at leading order in λ and one has to go a step further in the expansion to get a non trivial result. Noting that $A = F_R(0)$ and $B = G_R^{(2)}(0)$ one finds that the IR part of the integral has the form:

$$-\widetilde{g}^2 \lambda^2 F_R(0)^2 G_R^{(2)}(0) \int^{q < q_0/\lambda} \frac{d^4q}{(2\pi)^4} \left(1 - \frac{(k.q)^2}{k^2 q^2} \right) \left[\frac{1}{(q-k)^2} - \frac{1}{(q-\kappa k)^2} \right] = \qquad (2.6)$$

$$-\widetilde{g}^2 \frac{1}{64\pi^2} \lambda^2 k^2 (1 - \kappa^2) \log(q_0/(\lambda \, k)) F_R(0)^2 G_R^{(2)}(0) + \cdots$$

where the dots denote subleading $\mathcal{O}(k^2)$ terms. It still dominates over the UV part, although now it is only by a logarithm. We then write consistently $F_R(k^2) = a + bk^2\log(1/k^2) + \mathcal{O}(k^2)$ in the l.h.s, and we find:

$$F_R(k^2) = F_R(0)\left(1 - \widetilde{g}^2 \frac{1}{64\pi^2} F_R(0)^2 G_R^{(2)}(0) k^2 \log(M^2/k^2)\right) \qquad (2.7)$$

M^2 being some scale which we cannot derive from this IR expansion. Let us stress that equations (2.6) and (2.7) are meaningful only to the extent that $G_R^{(2)}(0)$ is finite and non zero, i.e. that $\alpha_G = 1$.

3. Numerical solution of the Schwinger-Dyson equation for the ghost

In this section we want to see whether the two types of solutions ($\alpha_F = 0$ and $2\alpha_F + \alpha_G = 0$) suggested by our analytical discussion in [5, 13] do actually exist for the same gluon propagator. We answer positively by solving numerically the ghost SD equation for given gluon propagator and vertex. In the following we shall use the subtracted form of the Schwinger-Dyson equation for the ghost in the Landau (i.e. Lorentz) gauge. This equation has been written above, eq. (1.12).

We start from an IR finite gluon propagator ($\alpha_G = 1$) extracted from our lattice data in pure Yang-Mills theory with Wilson gauge action, $\beta = 5.8$ and a lattice volume equal to 32^4, for momenta lower than $1.5\,\text{GeV}$; this choice is justified to have moderate UV artefacts. We extend it to larger momenta using a one loop asymptotic expansion (with $\Lambda_{\text{MOM}} = 1\,\text{GeV}$ corresponding to the standard $\Lambda_{\overline{MS}} = .240\,\text{GeV}$ of quenched lattice QCD). On the other hand, we take $H_1(q, k)$ to be constant with respect to both momenta.[9] As we said above, this is suggested by the lattice data for $q = k$ (i.e. for zero gluon momentum), but we extend it to all values of q and k. The authors of ref. [10] find a bare vertex very close to 1 in this zero momentum gluon configuration for a large range of $\sqrt{q^2}$.

We work in the MOM scheme, and set k'^2 appearing in eq. (1.12) as the squared renormalisation scale μ^2 (μ has been chosen at an optimum $1.5\,\text{GeV}$, not too high to allow the lattice data to be safe, and not too small to display the differences between solutions at small momenta). The equation we have to solve is:

$$\frac{1}{F_R(k^2)} = 1 - \widetilde{g}^2 \int \frac{d^4q}{(2\pi)^4}\left(1 - \frac{(k.q)^2}{k^2 q^2}\right) \times$$
$$\times \left[\frac{G_R((q-k)^2)}{((q-k)^2)^2} - \frac{G_R((q-k')^2)}{((q-k')^2)^2}\right] F_R(q^2)\Bigg|_{k'^2 = \mu^2}. \qquad (3.1)$$

Note that this equation implies that $\widetilde{g}^2$ depends only on the renormalisation point chosen for the propagators; it is independent of the particular way used to define the renormalisation of the vertex, in agreement with eq. (2.1). Eq. (3.1) can be transformed further to a new form which makes the numerical calculation and the presentation of the various solutions

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[9]This cannot be an exact statement, as already shown in perturbation by the calculations of ref. [6, 22]: although finite, the vertex invariants do depend on the momenta through the running α_s.

easier; for this, we subtract the equation at $k = 0$ to let the value of $F_R(k)$ at the origin appear and to eliminate the reference to the particular normalisation point μ, and we redefine also the unknown function to be calculated as $\widetilde{F}(k) = \widetilde{g}F_R(k)$. Then the reference to the value of $\widetilde{g}$ also disappears; we end with:

$$\frac{1}{\widetilde{F}(k^2)} = \frac{1}{\widetilde{F}(0)} - \int \frac{d^4q}{(2\pi)^4} \left(1 - \frac{(k.q)^2}{k^2q^2}\right) \times$$
$$\times \left[\frac{G_R((q-k)^2)}{((q-k)^2)^2} - \frac{G_R((q)^2)}{((q)^2)^2}\right] \widetilde{F}(q^2). \tag{3.2}$$

We solve equation (3.2) for $\widetilde{F}(k^2)$, for a set of values of $\widetilde{F}(0)$. It is easy to see that from this solution we can reconstruct the desired solution of eq. (3.1) for any renormalisation point and any value of $\widetilde{g}$. Indeed we have $\widetilde{g}(\mu) = \widetilde{F}(\mu^2)$ so that, for given μ and $\widetilde{g}$, it suffices to identify the value of $\widetilde{F}(0)$ such that $\widetilde{F}(\mu^2) = \widetilde{g}$. Then we reconstruct $F_R(k^2)$ through $F_R(k^2) = \widetilde{F}(k^2)/\widetilde{g}(\mu)$

By construction all the solutions found in this way are finite at the origin. The solution which diverges at the origin will be found by setting $\frac{1}{\widetilde{F}(0)} = 0$ in eq. (3.2). Alternatively it can also be approached by making $\widetilde{F}(0)$ larger and larger.

From the practical point of view, we have looked for solutions of eq. (3.2) with the integral cut in the UV at $q = 30\,\mathrm{GeV}$. We have discretized it in k and q. Taking values of the momenta spaced out by $0.01\,\mathrm{GeV}$ for $q \leq 2\,\mathrm{GeV}$ and by $0.1\,\mathrm{GeV}$ for $q \geq 2\,\mathrm{GeV}$ we have computed the angular integral of the r.h.s. of eq. (3.1). Then, we solved this equation by iteration. Minus the integral in the r.h.s. is positive, allowing an easy convergence. We linearize it at each step, following the Newton method, to accelerate the convergence of the iteration procedure, as suggested by Bloch.

The results are the following:

1) **Critical case, singular solution.** We find a solution with $\frac{1}{\widetilde{F}(0)} = 0$, i.e. $\widetilde{F}(0) = \infty$. We find then the corresponding "critical" constant:

$$\widetilde{g_c}^2 = \widetilde{F}(1.5 \text{ GeV}) = 33.198\ldots. \tag{3.3}$$

This value happens to be very close to the one expected from eq. (2.5):

$$\widetilde{g_c}^2 A^2 G^{(2)}(0)\frac{1}{10\pi^2} \approx 0.994\ldots. \tag{3.4}$$

The integration near $k = 0$ can be improved by taking explicitly into account the analytical behavior of the kernel, and assuming that the solution behaves as $1/k$ at small k. This procedure enforces eq. (2.5) and one checks actually that $\widetilde{g_c}^2 k^2 F(k^2)^2 G^{(2)}(k^2)\frac{1}{10\pi^2}$ goes very smoothly to 1 when $k \to 0$.

2) **Regular case.** We find a solution for all $\widetilde{F}(0) > 0$, and only one for each $\widetilde{F}(0) > 0$ with our method of solution. From our numerical solution at $\widetilde{g}^2 \simeq 29$, which corresponds to the best description of lattice data (see figure 2), we can test the
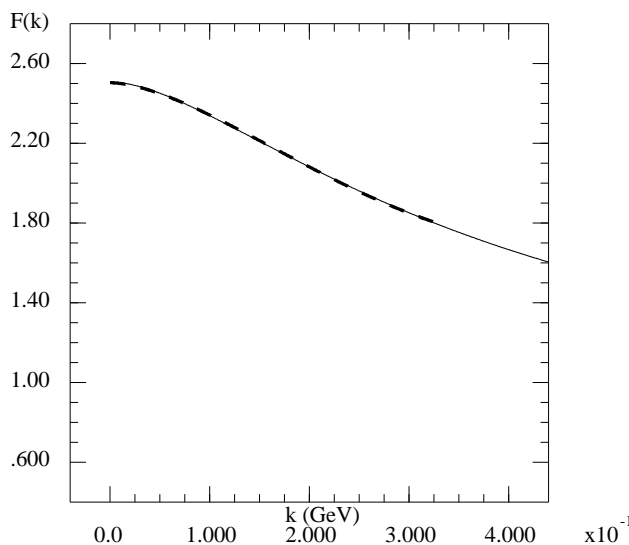
Figure 1: The $a + bk^2 \log(k^2)$ fit at small momentum (dashed line) to our continuum SD prediction for the ghost dressing function, renormalised at $\mu = 1.5\,\mathrm{GeV}$ for $\widetilde{g}^2 = 29$.(solid line) ; the slope of the $k^2 \log(k^2)$ term is 4.06; the agreement with the expected coefficient of $k^2 \log(k^2)$, 4.11 from the eq. (2.7), is striking.

relation (2.7) giving the $k^2 \log(k^2)$ term. The result is presented in figure 1 and the slope agrees well with what is expected from (2.7): 4.06 against 4.11.

The critical value of the coupling constant as well as the corresponding curve of $\widetilde{F}(k)$ can be very well approximated by the regular solutions at very large $\widetilde{F}(0)$. When $\widetilde{F}(0)$ becomes larger and larger eq. (2.7) remains valid only in a smaller and smaller region near $k = 0$, while in an intermediate region a $1/k$ behaviour is observed.

In conclusion, in the case $\alpha_G = 1$ we have exhibited a continuum set of IR finite solutions for arbitrary $F(0)$, and a unique singular solution for $\widetilde{g}^2 = \widetilde{g}_c^2$, with $\alpha_F = -\frac{1}{2}$. These are the only solutions obtained with our iteration procedure using the Newton method. Of course this doesn't prove that no other solution exists.

3.1 Why the regular solutions have not been obtained previously

The regular solutions could not be obtained by the proponents of the equation 1.3, because, as it seems to us, they discard them from the beginning and thereby choose the critical value of the coupling constant, by making an implicit assumption when solving the so-called "infrared equation" for the ghost SD equation. One can see this in the papers by R. Alkofer et al. (for instance [1], eqs. (43), (44)), or in the detailed discussion of Bloch [3]), eqs. (55) to (58) .

Let us explain this briefly. They consider the above unsubtracted equation (note that this requires then an UV cutoff, which we avoid in our discussions by considering the

subtracted form — see below, next section); we write again the unsubtracted form:

$$\frac{1}{F_R(k^2)} = \widetilde{Z}_3 - N_c g_R^2 \widetilde{z}_1 \int \frac{d^4q}{(2\pi)^4} \left(1 - \frac{(k.q)^2}{k^2 q^2}\right) \times$$
$$\times \left[\frac{G_R((q-k)^2)H_{1R}(q,k)}{((q-k)^2)^2}\right] F_R(q^2). \qquad (3.5)$$

One tries to match the small k^2 behaviour of the two sides of eq. (3.5). This is done for example in eq. (58) of [3]. To this end a condition is then written which consists in equating the coefficient of $(k^2)^{-\alpha_F}$ in the l.h.s with the corresponding one on the right. However, one notices that in the r.h.s. there is a constant contribution $\propto (k^2)^0$. Therefore, unless this constant term, $\tilde{Z}_3$, is cancelled by the integral contribution for $k \to 0$, we have necessarily $\alpha_F = 0$. To have $\alpha_F < 0$ as the author finds, one needs this cancellation. This is what is **implicitly assumed**, but not stated explicitly. The condition of cancellation is:

$$\widetilde{Z}_3 = N_c g_R^2 \widetilde{z}_1 \int \frac{d^4q}{(2\pi)^4} \left(1 - \frac{(k.q)^2}{k^2 q^2}\right) F_R(q^2) \Bigg|_{k=0}. \qquad (3.6)$$

The important point is that this additional equation **is not a consequence** of the starting SD ghost equation and, indeed, it is not fulfilled in general by the solutions of this basic equation, as we show by exhibiting actually IR finite solutions. In fact, it can be valid only for a **particular value of the coupling constant, the critical one** which is solution to the equation of Bloch, his eq. (58), and which we derive rigorously through the subtracted equation (see our eq. (2.5)).

Let us make precise that at this stage the value of the coupling constant is taken as a free parameter. However it should be fixed at the end to ensure consistency with the lattice data which we are using. This is what will be done in the next section.

4. Phenomenology

Having presented the general study and found the announced two types of solutions for the ghost dressing function, either regular or singular, we are facing the question: which one is effectively realised on the lattice, and therefore in true QCD? Let us recall the classical problems which hamper the answer: it is not possible to know from the lattice data with total certitude whether the ghost dressing function is singular or not, because 1) on the one hand, the "singular" qualification by itself does not tell how close one should be to the zero momentum for the singularity to show up and 2) on the other hand, one cannot get arbitrarily close to zero momentum on the lattice.

A better way is offered by our calculation: it predicts the behavior of the respective solutions for the ghost over the **whole range** of momentum and not only very close to $k = 0$; then looking to the lattice data, we can identify which one is the most compatible with the data.[10] From figure 2, we see that we can discard rather safely the singular solution, because it is passing much above the lattice data points over a **very large range**

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[10]At this stage, it is useful to stress the advantage of working with the renormalised form of the SD equations; indeed the continuum and lattice versions are more directly comparable than the bare ones. As
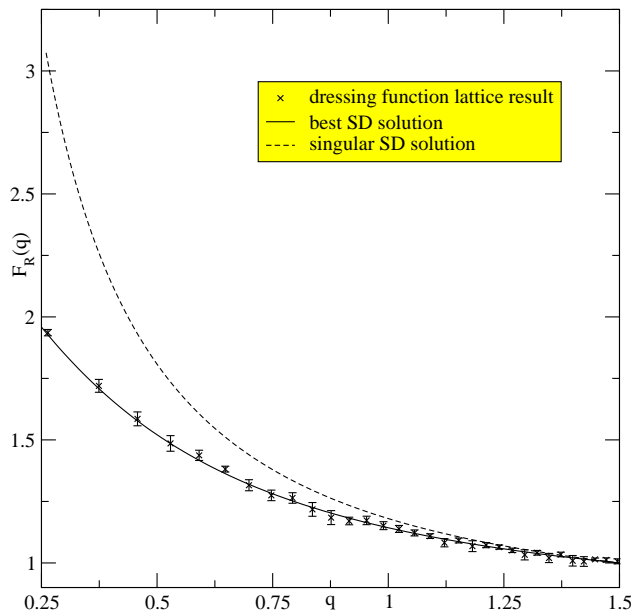
Figure 2: Comparison between the lattice SU(3) data at $\beta = 5.8$ and with a volume 32^4 for the ghost dressing function and our continuum SD prediction renormalised at $\mu = 1.5\,\text{GeV}$ for $\widetilde{g}^2 = 29$. (solid line); the agreement is striking; also shown is the singular solution which exists only at $\widetilde{g}^2 = 33.198\ldots$ (broken line), and which is obviously excluded.

of momentum: around 50% at the leftmost point which was measured on our lattice, $k = 0.26\,\text{GeV}$, but still quite sizeably near $k = 0.5\,\text{GeV}$. It is therefore quite unprobable that any lattice IR artefact could fill the gap. The advantage of our method is that, by calculating what the critical solution should be at rather large momenta, we are able to discard it more convincingly.

On the other hand, we find a very good description of the lattice data in the range $\widetilde{g}^2 = 28.3 - 29.8$ (the range is defined by one standard deviation except for the lowest point). This striking agreement is illustrated by figure 2 (for indication, we quote the IR limit $F_R(0) = 2.51$ for the same $\mu = 1.5$ and $\widetilde{g}^2 = 29$).

Moreover, we can perform the following consistency test. Starting from the equation (2.1) which defines $\widetilde{g}^2$ in terms of bare quantities we apply it to connect the continuum $\widetilde{g}^2$ to the lattice bare quantities:

$$\widetilde{g}^2 = N_c g_R^2 \widetilde{z}_1 = N_c(6/\beta)\ F_B^2(\mu^2)G_B(\mu^2)H_{1B}. \tag{4.1}$$

We then ask whether our range $\widetilde{g}^2 = 28.3 - 29.8$ is reasonably consistent with the r.h.s. of eq. (4.1) as given by lattice data. Let us stress that eq. (4.1) should be then only approximate in several respects; first, it is valid up to finite cutoff effects, as well as volume

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we have seen in our paper [5], the bare lattice equation for the ghost is affected by an important artefact which vanishes only very slowly with the cutoff, being of order $\mathcal{O}(g^2)$. In the renormalised version, this effect is included in the renormalisation constant $\widetilde{Z}_3$, and we are left only with the much smaller cutoff effects of the type $\mathcal{O}(a^n)$.

effects ; second, we have replaced the lattice vertex invariant $H_{1B}(q,k)$ by the constant H_{1B} (which is very rough) and we cannot test on the lattice the assumed constancy over the momenta which are actually implied in our calculation since we have only at our disposal a lattice measure of H_{1B} at $q = k$; last but not least, a lattice measurement of such vertex quantities is difficult because it is very noisy. Thus the test is only qualitative; nevertheless, the result is very encouraging as we see now.

Indeed, from the above value of $\tilde{g}^2$ found in the continuum on the one hand and the lattice data $\beta = 5.8$, $G_B(\mu^2) \simeq 2.89$ and $F_B(\mu^2) \simeq 1.64$ (μ is here chosen as $1.5\,\mathrm{GeV}$) on the other hand, we can deduce the value of the factor H_{1B} needed to satisfy equation (4.1) and which represents some average on momenta. We find $H_{1B} \simeq 1.2$. This number should be compared to the lattice measurements which are for H_{1B} at $q = k$, and which give about 1., with large errors (slightly larger or equal to 1., depending on the k value, see ref. [10]). The comparison seems very encouraging in view of the large uncertainties of the procedure: lattice artefacts, errors on H_{1B}, and finally the fact that our H_{1B} is some average on momenta away from $q = k$.

Another way of presenting the striking difference between the regular solution and the singular one is in terms of the familiar product discussed in the introduction: $G_R(k)F_R(k)^2$.[11] From the analytical discussion, in the **critical** case it should tend to $10\,\pi^2/\tilde{g}_c^2 \simeq 3.$ when $k \to 0$, and numerically it should be 3.14 at our smallest lattice momentum $k = 0.26$. This is completely at odds with the lattice data: the lattice value is 1.28 at $k = 0.26$ and there is a clear tendency towards still lower values at smaller k. On the contrary, our regular solution fits perfectly the lattice data. We illustrate this in figure 3. Let us stress that our SD solutions (continuous curve) are obtained **in the continuum** and **in infinite volume**; they appeal to the lattice data only to have a physically reasonable definite gluon propagator as an input to the SD equation.

5. Conclusion

The relation $\alpha_G + 2\alpha_F = 0$ is usually believed to be an unavoidable consequence of SD equations. The problem is that the lattice data grossly contradict it. Indeed, for small momenta $G(k^2)F(k^2)^2$ goes very fast to 0. We resolve this contradiction in the following way. We show that the mentioned belief is wrong and that an alternative exists by solving numerically the ghost SD equation with input from the lattice for the gluon propagator and the vertex.

The alternative is the one we have previously envisaged as a possibility in a general analytical analysis [5, 13]: $\alpha_F = 0$ with $\alpha_G \geqslant 1$, allowing $G(k^2)F(k^2)^2$ to go to 0 as shown by the data, but in the present article its existence is demonstrated by actually solving for F the equation for a given G. This solution violates the statement $\alpha_G + 2\alpha_F = 0$.

The relation $\alpha_G + 2\alpha_F = 0$ would imply that the ghost form-factor is singular, since $\alpha_G > 0$. The numerical solution of the equation then adds another strong reason for

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[11]Usually, this quantity is presented with multiplication by an additional factor including the renormalised coupling constant and possibly other factors. Here, we present the raw product to avoid any ambiguity in such procedures.
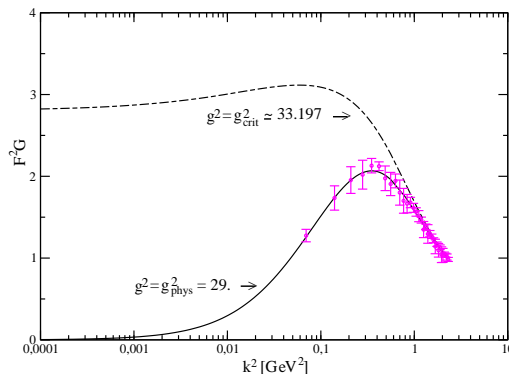
Figure 3: Comparison between our lattice SU(3) data at $\beta = 5.8$ and with a volume 32^4 for the product of gluon times square ghost dressing functions $G_R(k)F_R(k)^2$, renormalised at $\mu = 1.5\,\text{GeV}$, and the corresponding curve for the continuum singular solution $\alpha_G + 2\alpha_F = 0$, which exists only at $\widetilde{g}^2 = 33.198\dots$; the disagreement is manifest. Up to a factor $g^2/4\pi$, it corresponds to the $\alpha(k^2)$ presented in figure 8 by Fischer [23] and the form is actually very similar. Also shown is our continuum regular solution for $\widetilde{g}^2 = 29.$ (solid line) for which the agreement is striking.

rejecting this relation. A singular solution — which necessarily satisfies $\alpha_G + 2\alpha_F = 0$ — only exists for a definite value of the coupling constant. We calculate it and find that it differs grossly from the lattice data on a large range of k, and not only for the smallest momenta.

The alternative solution, which is regular in the infrared ($\alpha_F = 0$), is realised if the coupling constant is smaller than a certain critical value while the singular one is present only at the critical value. The lattice data for the ghost are very well reproduced for a coupling constant close to the one expected from the value $\beta = 6.0$ used for the lattice calculation. We are then confident to have found the actual QCD solution, up to moderate artefacts.

Our numerical study therefore adds strong new arguments from the lattice data in favor of this alternative. It is based on the ghost equation only, since we feel that the gluon equation, being much more complicated, suffers from much more uncertainties due to the necessary critical approximations to be made - this has been illustrated by the findings of Bloch [3].

The alternative, regular, solution has not been found in usual studies, because they have chosen by construction the critical value.

The important physical consequence is that we do not get the alleged non trivial IR fixed point for the MOM coupling constant since $g_R(k) \to 0$ when $k \to 0$ at fixed g_0 as for the three-gluon couplings, in agreement with lattice data. At this point, it is important to insist on the fact that there are infinitely many definitions of "the" QCD coupling constant. A priori, there is no reason for **their IR behaviour to be universal**. In particular, the

ones defined from elementary fields Green functions have no reason to behave in the same way as more physical definitions such as taken from the perturbative expansion of certain physical hadronic amplitudes.

Acknowledgments

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